Are monochromatic Pythagorean triples avoidable?

S. Eliahou, J. Fromentin, V. Marion-Poty, D. Robilliard

Abstract

A Pythagorean triple is a triple of positive integers $a, b, c \in \mathbb{N}_+$ satisfying $a^2 + b^2 = c^2$. Is it true that, for any finite coloring of $\mathbb{N}_+$, at least one monochromatic Pythagorean triple must be monochromatic? In other words, is the Diophantine equation $X^2 + Y^2 = Z^2$ *regular*? This problem has been open since several decades, even restricted to 2-colorings. In this note, we introduce *partial morphisms*, which are special colorings of $\mathbb{N}_+$ by finite groups with partly multiplicative properties. We show that, for many such colorings, monochromatic Pythagorean triples are impossible to avoid in the integer interval $[1, 10000]$.

Keywords: Pythagorean triple; partition-regular equation; partial morphism; SAT solver.

1 Introduction

A triple (a, b, c) of positive integers is a *Pythagorean triple* if it satisfies $a^2 + b^2 = c^2$, as $(3, 4, 5)$ for instance. Is it possible to find a finite coloring of the set of positive integers in such a way that no Pythagorean triple will be monochromatic? While this typical Ramsey-type question has been open for decades [4], there is no consensual conjecture as to whether the answer should be positive or not [6]. For background on Ramsey theory, see [7].

In this paper, we tackle this problem by restricting the considered colorings to special ones satisfying certain algebraic properties. In all cases reported here, monochromatic Pythagorean triples turn out to be unavoidable. Whether this is the case for arbitrary finite colorings remains to be seen.

In another direction and with a completely different approach, J. Cooper and R. Overstreet recently obtained, remarkably, a 2-coloring of the integer interval $[1,7664]$ avoiding monochromatic Pythagorean triples in that interval [3]. They achieved this by translating the problem into a system of logical constraints to be then fed to a SAT solver. See also [2] for a related earlier work.

At any rate, the work presented here leads us to conjecture that the answer to the question in the title is, in fact, negative.

2 Notation and background

We shall denote by $\mathbb{N}$ the set of nonnegative integers, by $\mathbb{N}_+ = \{n \in \mathbb{N} \mid n \geq 1\}$ the subset of positive integers, and by $\mathbb{P} = \{2,3,5,7,11,13,\dots\}$ the subset of prime numbers. Given positive integers $a \leq b$, we shall denote the integer interval they bound by

$$[a,b] \ = \ \{c \in \mathbb{N} \mid a \leq c \leq b\}.$$

Definition 2.1. *A* Pythagorean triple *is a triple* (a,b,c) *of positive integers satisfying* $a^2+b^2=c^2$. *Such a triple is said to be* primitive *if it satisfies* $\gcd(a,b,c)=1$.

Obviously, since the equation $X^2+Y^2=Z^2$ is homogeneous, every Pythagorean triple is a scalar multiple of a primitive one.

The parametrization of primitive Pythagorean triples is well known. Indeed, *every* primitive Pythagorean triple is of the form

$$(m^2-n^2, 2mn, m^2+n^2),$$

where m,n are coprime positive integers such that $m-n$ is positive and odd.

Following Rado [8], a Diophantine equation $f(X_1,\dots,X_n)=0$ is said to be *partition-regular*, or *regular* for short, if for every finite coloring of $\mathbb{N}_+$, there is a monochromatic solution $(x_1,\dots,x_n) \in \mathbb{N}_+^n$ to it. More specifically, for given $k \in \mathbb{N}_+$, the equation is said to be *k-regular* if, for every k-coloring of $\mathbb{N}_+$, there is a monochromatic solution to it. Note that regularity is equivalent to k-regularity for all $k \in \mathbb{N}_+$, and that k-regular implies $(k-1)$-regular if $k \geq 2$.

With this terminology, the question under study here is to determine whether the Diophantine equation $X^2+Y^2-Z^2=0$ is regular or not. And if not, whether it is at least 2-regular or 3-regular. This problem is open since several decades.

3 Standard and partial morphisms

In this paper, we introduce colorings of $\mathbb{N}_+$ by a finite additive group G and satisfying special algebraic properties. Below, we shall mainly focus on the groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

To start with, we consider monoid morphisms in the usual sense, i.e. maps $f\colon \mathbb{N}_+ \to G$ satisfying

$$f(xy) = f(x) + f(y)$$

for all $x, y \in \mathbb{N}_+$. Note that such a morphism is *completely and freely determined by its values* $\{f(p)\}_{p\in\mathbb{P}}$ *on the prime numbers*.

These morphisms f are particularly interesting in the present context, since if a primitive Pythagorean triple (a,b,c) fails to be monochromatic under f, then the same holds for all its scalar multiples (ad,bd,cd) with $d \in \mathbb{N}_+$. Indeed, if $f(x) \neq f(y)$, then $f(xd) \neq f(yd)$ for all $d \geq 1$. This follows from the property $f(zd) = f(z) + f(d)$ for all $z \in \mathbb{N}_+$ and the fact that the values lie in a group.

We shall prove in a subsequent section that, for any morphism $f\colon \mathbb{N}_+ \to G$ where G is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$, monochromatic primitive Pythagorean triples are unavoidable.

Consequently, the constraints of morphisms must be somewhat relaxed if we want to try and find, if at all possible, a 2-coloring avoiding monochromatic Pythagorean triples. On the other hand, some structure on the considered colorings is needed, in order to have a more manageable function space size. This prompts us to consider maps $f\colon \mathbb{N}_+ \to G$ satisfying weaker conditions than morphisms, and which we now define.

First, for any positive integer n, we denote by $\operatorname{supp}(n)$ the set of prime factors of n. For instance, $\operatorname{supp}(60) = \{2,3,5\}$.

Definition 3.1. *Let $(G,+)$ be an abelian group. Let $\mathbb{P}_0 \subseteq \mathbb{P}$ be a given subset of the prime numbers. We say that a map*

$$f\colon \mathbb{N}_+ \to G$$

is a $\mathbb{P}_0$-partial morphism*, or a* $\mathbb{P}_0$-morphism *for short, if the following properties hold. For any $n \in \mathbb{N}_+$, let $n_0 \in \mathbb{N}_+$ be the largest factor of n such that $supp(n_0) \subseteq \mathbb{P}_0$ and let $n_1 = n/n_0$, so that $n = n_0 n_1$. Then*

- $f(n) = f(n_0) + f(n_1)$*;*

- $f(n_1) = f(a) + f(b)$ *for any coprime integers* a, b *satisfying* $n_1 = ab$.

An equivalent way of expressing this notion is as follows. For any $n \in \mathbb{N}_+$, consider its unique prime factorization

$$n = \prod_{p \in \mathbb{P}} p^{\nu_p(n)}$$

where $\nu_p(n) \in \mathbb{N}$ for all p. Then, the map $f \colon \mathbb{N}_+ \to G$ is a $\mathbb{P}_0$-morphism if for all $n \in \mathbb{N}_+$, we have

$$f(n) = f\Big(\prod_{p \in \mathbb{P}_0} p^{\nu_p(n)}\Big) + \sum_{p \notin \mathbb{P}_0} f\big(p^{\nu_p(n)}\big).$$

Thus, a $\mathbb{P}_0$-morphism is entirely and freely determined by its values on the set of positive integers

$$S(\mathbb{P}_0) = \big\{n_0 \in \mathbb{N}_+ \mid \mathrm{supp}(n_0) \subseteq \mathbb{P}_0\big\} \bigsqcup \big\{p^\nu \mid p \in \mathbb{P} \setminus \mathbb{P}_0, \nu \in \mathbb{N}_+\big\}.$$

For instance, any $\{2,3\}$-morphism is freely determined by its values on the integers of the form $2^a 3^b$ or p^c with $p \in \mathbb{P}$, $p \geq 5$, where $a, b, c \in \mathbb{N}$ and $a + b \geq 1$, $c \geq 1$.

Remark 3.2. *Here are a few easy observations about* $\mathbb{P}_0$*-morphisms* $f \colon \mathbb{N}_+ \to G$.

- *If* $\mathbb{P}_0 = \emptyset$, *then* f *is characterized by the property* $f(xy) = f(x) + f(y)$ *for all coprime positive integers* x, y.

- *On the other side of the spectrum, if* $\mathbb{P}_0 = \mathbb{P}$, *then* f *is just a set-theoretical map without any special property or structure.*

- *More generally, if* $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \mathbb{P}$, *then any* $\mathbb{P}_0$*-morphism is also a* $\mathbb{P}_1$*-morphism.*

Note finally that any usual morphism is a $\emptyset$-morphism.

4 Coloring by morphisms

The interest of using morphisms $f \colon \mathbb{N}_+ \to G$ as coloring functions is that such a coloring admits a monochromatic Pythagorean triple if and only if it admits a monochromatic *primitive* Pythagorean triple. This is why we only need consider primitive Pythagorean triples in this section.

Proposition 4.1. *For any morphism $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$, monochromatic primitive Pythagorean triples are unavoidable. More precisely, such monochromatic triples are already unavoidable in the integer interval* $[1,533]$. *And finally, 533 is minimal with respect to this property.*

The proof below relies on some computer assistance but, with patience, everything can be checked by hand.

Proof. Let $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$ be any morphism. Then f is determined by its values on the prime numbers via the formula

$$f(n) = f\Big(\prod_{p\in\mathbb{P}} p^{\nu_p(n)}\Big) = \sum_{p\in\mathbb{P}} \nu_p(n) f(p)$$

for any $n \in \mathbb{N}_+$. Plainly, the only primes p which actually contribute to the value of $f(n)$ are those for which $\nu_p(n)$ is odd. For example, we have $f(12) = f(3)$.

For $n \in \mathbb{N}_+$, let us denote by oddsupp(n) the *odd support* of n, i.e. the set of primes p for which $\nu_p(n)$ is odd. Thus, the above formula for $f(n)$ reduces to

$$f(n) = \sum_p f(p),$$

where p runs through oddsupp(n) only.

We shall restrict our attention to the 13 first primes, denoted $p_1,\dots,p_{13}$ in increasing order, and shall denote their set by $\mathbb{P}_{13}$. Thus $\mathbb{P}_{13} = \{2,3,\dots,37,41\}$. Further, let us set

$$\mathbb{N}_{|\mathbb{P}_{13}} = \{n \in \mathbb{N}_+ \mid \text{oddsupp}(n) \subseteq \mathbb{P}_{13}\}.$$

For instance, the first few positive integers *not* in $\mathbb{N}_{|\mathbb{P}_{13}}$, besides the primes $p \geq 43$, are 86, 94, 106, 118, 122, etc.

By the above formula, the value of $f(n)$ for any $n \in \mathbb{N}_{|\mathbb{P}_{13}}$ is entirely determined by the length 13 binary vector

$$v(f) = \big(f(p_1),\dots,f(p_{13})\big) \in (\mathbb{Z}/2\mathbb{Z})^{13}.$$

Consider now the set $\mathcal{T}$ of all primitive Pythagorean triples in the integer interval $[1,532]$. There are 84 of them, the lexicographically last one being $\{279,440,521\}$. Among them, we shall distinguish the subset $\mathcal{T}_{13}$ defined as

$$\mathcal{T}_{13} = \big\{(a,b,c) \in \mathcal{T} \mid a,b,c \in \mathbb{N}_{|\mathbb{P}_{13}}\big\},$$

i.e. those triples in $\mathcal{T}$ whose three elements have odd support in $\mathbb{P}_{13}$. One finds that $|\mathcal{T}_{13}| = 32$. For definiteness, here is this set:

$\mathcal{T}_{13} = \big\{\{3,4,5\}, \{5,12,13\}, \{8,15,17\}, \{7,24,25\}, \{20,21,29\}, \{12,35,37\}, \{9,40,41\},$
$\{33,56,65\}, \{16,63,65\}, \{13,84,85\}, \{36,77,85\}, \{44,117,125\}, \{17,144,145\}, \{24,143,145\},$
$\{119,120,169\}, \{57,176,185\}, \{104,153,185\}, \{133,156,205\}, \{84,187,205\}, \{21,220,221\},$
$\{140,171,221\}, \{161,240,289\}, \{204,253,325\}, \{36,323,325\}, \{135,352,377\}, \{152,345,377\},$
$\{87,416,425\}, \{297,304,425\}, \{31,480,481\}, \{319,360,481\}, \{155,468,493\}, \{132,475,493\}\big\}.$

Perhaps surprisingly, it turns out that there are exactly two *avoiding morphisms*

$$f_1, f_2 \colon \mathbb{N}_{|\mathbb{P}_{13}} \to \mathbb{Z}/2\mathbb{Z}$$

for which no $(a,b,c) \in \mathcal{T}_{13}$ is monochromatic. They are determined by the length 13 binary vectors $v(f_1) = w_1$, $v(f_2) = w_2$, where

$$\begin{aligned} w_1 &= 0101111101001, \\ w_2 &= 0101111111001. \end{aligned}$$

Note that w_1, w_2 only differ at the 9th digit.

Now, the 85th primitive Pythagorean triple is $(308, 435, 533)$. As it happens, that triple is mapped to $(0,0,0)$ by both f_1 and f_2. Indeed, the prime factorizations of 308, 435 and 533 only involve the primes

$$\begin{array}{llll} p_1 = 2, & p_2 = 3, & p_3 = 5, & p_4 = 7, \\ p_5 = 11, & p_6 = 13, & p_{10} = 29, & p_{13} = 41, \end{array}$$

and are the following: $308 = p_1^2 p_4 p_5$, $435 = p_2 p_3 p_{10}$, $533 = p_6 p_{13}$. Hence, for $f = f_1$ or f_2, we have

$$\begin{array}{lllllll} f(308) &=& f(p_4) + f(p_5) &=& 1+1 &=& 0, \\ f(435) &=& f(p_2) + f(p_3) + f(p_{10}) &=& 1+0+1 &=& 0, \\ f(533) &=& f(p_6) + f(p_{13}) &=& 1+1 &=& 0. \end{array}$$

We conclude, as claimed, that for every morphism $g \colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$, there must be a primitive Pythagorean triple in $[1, 533]$ which is monochromatic under g.

The fact that 533 is minimal with respect to this property is witnessed by the existence of many morphisms f under which none of the 84 primitive Pythagorean triples in $[1, 532]$ is monochromatic. The values on $p_1, \dots, p_{13}$ of these avoiding morphisms f must of course be specified by either w_1 or w_2, but there are further restrictions. Indeed, their values on all primes turn out to be constrained as follows: either

$$f(p_i) = \begin{cases} 0 & \text{for } i = 1,3,9,11,12,18,21,30,57,74,80,89, \\ 1 & \text{for } i = 2,4,5,6,7,8,10,13,16,24,26,55,65, \end{cases}$$

or

$$f(p_i) = \begin{cases} 0 & \text{for } i = 1,3,11,12,18,21,25,30,59,74,89, \\ 1 & \text{for } i = 2,4,5,6,7,8,9,24,26,55,65,70, \end{cases}$$

with complete freedom on all other primes. Note that the 13 first bits of the first and of the second type, i.e. the $f(p_i)$'s for $i \leq 13$, make up w_1 and w_2, respectively. □

An analogous result holds for morphic 3-colorings, established by an exhaustive computer search.

Proposition 4.2. *For any morphism* $f\colon \mathbb{N}_+ \to \mathbb{Z}/3\mathbb{Z}$*, monochromatic primitive Pythagorean triples are unavoidable. More precisely, at least one such triple in the integer interval* $[1,4633]$ *is monochromatic under* f*. And* 4633 *is minimal with respect to that property.*

Here is one particular morphism $f\colon \mathbb{N}_+ \to \mathbb{Z}/3\mathbb{Z}$ for which no Pythagorean triple in the interval $[1,4632]$ is monochromatic; it suffices to specify which primes in that interval are colored 1 or 2, the rest being colored 0. Denoting by p_i the ith prime for $i \geq 1$, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on, we set:

$$f(p_i) = \begin{cases} 1 & \text{if } i \in A, \\ 2 & \text{if } i \in \{6,7,23,24,29,30,33,74\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $A = \{1,2,5,11,12,13,16,17,19,20,21,25,37,45,55,65,68,70,71,82,84,89,98,112,123,130,135,151,189,198,203,220,245,267,345,355,359,381,401,443,464,514,561,583,610,612,624\}$. As said above, under this particular morphism, no Pythagorean triple in the interval $[1,4632]$ is monochromatic. For the record, there are 735 primitive Pythagorean triples in that interval.

5 Coloring by partial morphisms

We now turn to partial morphisms. Let us start first with an easy remark concerning primitive Pythagorean triples.

Remark 5.1. *There exist partial morphisms* $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$ *under which no* primitive *Pythagorean triple is monochromatic.*

One obvious example is the 2-coloring given by $f(n) = 1$ if n is even and 0 otherwise. Since any primitive Pythagorean triple (a,b,c) contains exactly one even number, it is not monochromatic under f.

Two similar examples arise by mapping multiples of 3 to color 1, or else multiples of 5 to color 1, and the rest to color 0, respectively. Indeed, any primitive Pythagorean triple (a,b,c) contains at least one multiple of 3, and one multiple of 5 as well; this easily follows from the fact that the only nonzero square mod 3 is 1, and the only nonzero squares mod 5 are ± 1. But since a,b,c are assumed to be coprime, and since $a^2+b^2=c^2$, they cannot be *all three* mapped to 1, or to 0, by these two 2-colorings.

These three 2-colorings are $\emptyset$-morphisms, as they satisfy $f(xy) = f(x)+f(y)$ for all coprime positive integers x,y. More precisely, they are characterized by the values

$$f(p_0^\nu) = 1,\ f(p^\nu) = 0$$

for all $\nu \geq 1$ and all primes $p \neq p_0$, where $p_0 = 2$, 3 or 5, respectively.

Perhaps surprisingly, for any $\emptyset$-morphism $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$, it turns out again that monochromatic Pythagorean triples are unavoidable. Here are even stronger results, obtained by exhaustive computer search with an algorithm briefly described below.

Proposition 5.2. *For any* $\mathbb{P}_0$*-morphism* $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$*, where* $\mathbb{P}_0$ *is one of the sets* $\{2,3,5\}$*,* $\{2,3,5,7\}$ *and* $\{2,3,5,7,11\}$*, monochromatic Pythagorean triples are unavoidable. More precisely, they are unavoidable in the integer interval* $[1,N]$*, where*

$$N = \begin{cases} 533 & \text{if} \quad \mathbb{P}_0 = \{2,3,5\}, \\ 565 & \text{if} \quad \mathbb{P}_0 = \{2,3,5,7\}, \\ 696 & \text{if} \quad \mathbb{P}_0 = \{2,3,5,7,11\}. \end{cases}$$

Moreover, in each case, N is minimal with respect to that property.

In the same vein, using the same algorithm, here is an attempt with the first seven prime numbers. The computer time needed to reach that statement was approximately 80 days on an Intel(R) Xeon(R) E5 at 2.20GHz. In contrast with the preceding statements, we do not know the exact threshold.

Proposition 5.3. *Let* $\mathbb{P}_0 = \{2,3,5,7,11,13,17\}$*. Then, for any* $\mathbb{P}_0$*-morphism* $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$*, monochromatic Pythagorean triples are unavoidable in the integer interval* $[1,9685]$.

5.1 The algorithm

Here is a brief description of the algorithm used. It consists of a recursive, backtracking search, that tries to color all elements in Pythagorean triples within a given integer interval $[1,M]$ without creating monochromatic such triples.

Let us consider a fixed subset $\mathbb{P}_0 \subset \mathbb{P}$. The set of variables is then the set $S(\mathbb{P}_0)$ of positive integers defined in Section 3, namely

$$S(\mathbb{P}_0) = \left\{n_0 \in \mathbb{N}_+ \mid \mathrm{supp}(n_0) \subseteq \mathbb{P}_0\right\} \bigsqcup \left\{p^\nu \mid p \in \mathbb{P} \setminus \mathbb{P}_0, \nu \in \mathbb{N}_+\right\}.$$

For $n \in \mathbb{N}_+$, let us denote by $\mathit{fact}(n) = \{q_1,\dots,q_k\}$ the unique subset of $S(\mathbb{P}_0)$ such that

$$n = \prod_{i=1}^{k} q_i$$

and where each $q_i \in S(\mathbb{P}_0)$ is *maximal*, in the sense that no proper multiple of q_i dividing n belongs to $S(\mathbb{P}_0)$. Thus, for a given $\mathbb{P}_0$-morphism $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$, and for $n \in \mathbb{N}_+$, we have

$$f(n) = \sum_{q\in \mathit{fact}(n)} f(q).$$

For instance, if $\mathbb{P}_0 = \{2,3,5\}$ and $n = 64680 = 2^3\cdot 3\cdot 5\cdot 7^2\cdot 11$, the maximal $S(\mathbb{P}_0)$-factors of n are $120 = 2^3\cdot 3\cdot 5$, $49 = 7^2$ and 11. Thus $\mathit{fact}(n) = \{120,49,11\}$, and $f(n) = f(120)+f(49)+f(11)$ for any $\mathbb{P}_0$-morphism f as above.

The algorithm will try to assign a suitable color in $\mathbb{Z}/2\mathbb{Z}$ to each variable, but the order in which this is done is important and may strongly affect the running time. To define a proper assignment order, we introduce the following notation. Given a positive integer M, let $\mathbb{T}_M$ denote the set of all Pythagorean triples contained in the integer interval $[1,M]$. Then, for $q \in S(\mathbb{P}_0)$ and $\{a,b,c\} \in \mathbb{T}_M$, we define

$$\delta_q^{\{a,b,c\}} = \begin{cases} 1 & \text{if } q \in \mathit{fact}(a) \cup \mathit{fact}(b) \cup \mathit{fact}(c), \\ 0 & \text{otherwise.} \end{cases}$$

The *weight* of the variable q is now defined as

$$w(q) = \sum_{t\in\mathbb{T}_M} \delta_q^t,$$

i.e. the number of Pythagorean triples in $[1,M]$ where q appears in the decomposition $\mathit{fact}(n)$ of one of the triple elements n. The variables are then ordered by decreasing weight, and the algorithm assigns a value in $\mathbb{Z}/2\mathbb{Z}$ to the variables in that order. Thus, variables constrained by the greatest number of triples in which they are involved as a maximal $S(\mathbb{P}_0)$-factor are tested first.

Once a variable is assigned, the algorithm performs forward arc checking [1], that is it computes, if possible, the color of all triple elements following the current partial morphism. This color computation of an element m is possible if all variables in $fact(m)$ are already colored as explained in Section 3. If the coloration of any member creates a monochromatic triple, then the algorithm tries the other color for the variable if any left, or backtracks.

5.2 The function $N(\mathbb{P}_0)$

The above results prompt us to introduce the following function.

Definition 5.4. *For any subset $\mathbb{P}_0 \subseteq \mathbb{P}$, we denote by $N(\mathbb{P}_0)$ the largest integer N if any, or ∞ otherwise, such that there exists a 2-coloring of the integer interval $[1,N]$ by a $\mathbb{P}_0$-morphism which avoids monochromatic Pythagorean triples in that interval.*

The above results may thus be expressed as follows:

$$\begin{aligned} N(\{2,3,5\}) &= 532, \\ N(\{2,3,5,7\}) &= 564, \\ N(\{2,3,5,7,11\}) &= 695, \\ N(\{2,3,5,7,11,13,17\}) &\leq 9684. \end{aligned}$$

Moreover, whether the equation $X^2+Y^2=Z^2$ is 2-regular or not is equivalent to whether $N(\mathbb{P})$ is finite or infinite, respectively. This follows from a standard compactness argument.

5.3 On small sets of primes up to 100

Finally, we consider $\mathbb{P}_0$-morphisms $f\colon \mathbb{N}_+ \to \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{P}_0$ ranges through all sets of prime numbers in $[2,100]$ of cardinality 3, 4 and 5. Note that $[2,100]$ contains 25 prime numbers. Needless to say, a considerable amount of code optimization and computer time were needed in order to establish the findings below.

5.3.1 The case $|\mathbb{P}_0| = 3$

There are $\binom{25}{3} = 2300$ sets of three distinct prime numbers smaller than 100.

Proposition 5.5. *Among the* 2300 *subsets* $\mathbb{P}_0 \subset \mathbb{P} \cap [2,100]$ *of cardinality 3, one has* $N(\mathbb{P}_0) = 532$ *for all but 29 of them. These 29 exceptions are as follows:*

$$N(\mathbb{P}_0) = \begin{cases} 544 & \textit{if } \mathbb{P}_0 \in \big\{\{2,3,13\},\{3,5,17\},\{5,13,41\}\big\}, \\ 564 & \textit{if } \mathbb{P}_0 \in \big\{\{2,3,7\}\big\} \cup \big\{\{7,11,a\} \mid a \in \mathbb{P} \cap [2,100] \setminus \{7,11\}\big\}, \\ 628 & \textit{if } \mathbb{P}_0 \in \big\{\{2,3,19\},\{3,13,19\}\big\}. \end{cases}$$

5.3.2 The case $|\mathbb{P}_0| = 4$

There are $\binom{25}{4} = 12650$ sets of four distinct prime numbers smaller than 100.

Proposition 5.6. *For all subsets* $\mathbb{P}_0 \subset \mathbb{P} \cap [2,100]$ *of cardinality 4, one has*

$$532 \le N(\mathbb{P}_0) \le 784$$

and, more precisely,

$$\begin{aligned} N(\mathbb{P}_0) \in \ & \{532,543,544,547,564,577,594,614,624,628, \\ & 649,656,662,666,679,688,696,739,778,784\}. \end{aligned}$$

Remark 5.7. *The value* 784 *above is attained only once, by* $\mathbb{P}_0 = \{3,5,19,23\}$. *The same holds for the next few largest values, including* $778 = N(\{3,13,19,23\})$ *and* $739 = N(\{2,3,19,23\})$. *All three cases involve the subset* $\{3,19,23\}$. *Compare with the most performant triples found in Proposition 5.5, namely* $\{2,3,19\}$ *and* $\{3,13,19\}$, *both containing* $\{3,19\}$.

Remark 5.8. *The behavior of* $N(\mathbb{P}_0)$ *may be quite subtle. For instance, one has* $N(\mathbb{P}_0) = 564$ *for all quadruples* $\mathbb{P}_0$ *in* $\mathbb{P} \cap [2,100]$ *containing the pair* $\{7,11\}$, *with one single exception given by* $N(\{7,11,13,17\}) = 624$.

5.3.3 The case $|\mathbb{P}_0| = 5$

There are $\binom{25}{5} = 53130$ sets of five distinct prime numbers smaller than 100.

Proposition 5.9. *For every subset* $\mathbb{P}_0 \subset \mathbb{P} \cap [2,100]$ *of cardinality 5, we have*

$$532 \leq N(\mathbb{P}_0) \leq 900.$$

Moreover, the only such subsets $\mathbb{P}_0$ *attaining the maximum* $N(\mathbb{P}_0) = 900$ *are*

$$\{2,3,7,19,23\}, \{2,3,17,19,23\}.$$

5.4 Summary

The findings of section 5.3 may be summarized as follows.

Proposition 5.10. *For every subset* $\mathbb{P}_0 \subset \mathbb{P} \cap [2,100]$ *of cardinality at most 5, and for every 2-coloring*

$$f\colon [1,901] \to \mathbb{Z}/2\mathbb{Z}$$

by a $\mathbb{P}_0$*-morphism, monochromatic Pythagorean triples are unavoidable.*

As for the $\binom{25}{6} = 177100$ subsets of cardinality 6 of $\mathbb{P} \cap [2,100]$, an exhaustive search cannot currently be completed in a reasonable amount of time. A first partial search has yielded 900 again as the highest value of $N(\mathbb{P}_0)$ found so far, achieved by the following subsets:

$$\{2,3,5,7,19,23\}, \{2,3,5,11,19,23\}, \{2,3,5,19,23,41\},$$
$$\{2,3,5,17,19,23\}, \{2,3,5,19,23,47\}, \{2,3,5,19,23,53\}.$$

Interestingly, looking more closely at the results of section 5.3, one notes that the subsets $\mathbb{P}_0 \subset \mathbb{P} \cap [2,100]$ maximizing the function N all contain $\{3,19\}$ if $|\mathbb{P}_0| = 3$, or $\{3,19,23\}$ if $|\mathbb{P}_0| = 4$, or $\{2,3,19,23\}$ if $|\mathbb{P}_0| = 5$. While the case $|\mathbb{P}_0| = 6$ is largely incomplete, an analogous statement might well hold.

As hinted to in the Introduction, the present work leads us to conjecture that monochromatic Pythagorean triples are unavoidable in $[1,10000]$ under any 2-coloring of that interval.

Acknowledgment. We thank Gyan Prakash for interesting information related to this work and for reference [5].

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