The Global Krylov subspace methods and Tikhonov regularization for image restoration

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1. Introduction

- The problem of image restoration consists of the reconstruction of an original image that has been digitized and has been degraded by a blur and an additive noise.

- This problem may be formulated by the following linear model (Andrews (1977), Jain (1989), ...)

\[
g(i,j) = (f * h)(i,j) + \nu(i,j) = \sum_{l,k} f(l,k)h(i-l,k-n) + \nu(i,j),
\]

where

- \( f \) represents the true image,
- \( h \) is the Point Spread Function (PSF),
- \( \nu \) is the additive noise
- \( g \) is the degraded image.

- Image restoration techniques apply an inverse procedure to obtain an estimate of the original image.
1. Introduction

- Using the Fourier transform, the problem of image restoration may be reformulated as follows

\[ \hat{g}(i,j) = \hat{h}(i,j) \cdot \hat{f}(i,j) + \hat{\nu}(i,j). \]  

- The problem may also be modeled in a matrix-vector form as

\[ g = Hx + n. \]  

- \( x = \text{vec}(X) \): true image \( X \), \( g = \text{vec}(G) \): distorted image \( G \) and \( n = \text{vec}(N) \) the noise of size \( n^2 \times 1 \).
- The \( \text{vec} \) operator transforms a matrix \( A \) of size \( n \times p \) to a vector \( a \) of size \( np \times 1 \) by stacking the columns of \( A \).
- \( H \) is obtained from the PSF and is called the PSF matrix (or blur matrix). The matrix \( H \) is of high size \( n^2 \times n^2 \) and is ill-conditioned.
- If the image is of size \( 512 \times 512 \), the matrix \( H \) is of size \( 262144 \times 262144 \).

- In practice, the PSF is usually not available.
- Both the blur matrix and the restored image must be performed from the observed noisy blurred image: blind image restoration.
2. Estimation of the point spread function

- **Blind restoration**: We need to estimate the matrix $P$ containing the image of the point spread function.
- The proposed method is based on the iterative deconvolution scheme introduced by Ayers and Dainty, 1977:
2. Estimation of the point spread function

\[ \hat{F}_k = \text{FFT}(f_k) \]
\[ \hat{H}_k = \text{FFT}(h_k) \]
\[ F_{k+1} = \hat{F}_k + \Delta F_k \]
\[ f_{k+1} = \text{IFFT}(F_{k+1}) \]
\[ P = H_k \]
\[ H_{k+1} = \hat{H}_k + \Delta H_k \]

Initial guests: \( f_k, h_k, k = 0 \)

Figure: Iterative PSF estimating algorithm
2. Estimation of the point spread function

Ayers & Dainty 1977

For \( k = 0, 1, 2, \ldots \),
\[
F_{k+1} = \hat{F}_k + \Delta F_k \quad \text{and} \quad H_{k+1} = \hat{H}_k + \Delta H_k
\]
with

\[
\Delta F_k = \frac{(G - \hat{F}_k \odot \hat{H}_k) \odot \hat{H}_k}{||\hat{H}_k||^2_F + \alpha^2}
\]

and

\[
\Delta H_k = \frac{(G - \hat{F}_k \odot \hat{H}_{k-1}) \odot \hat{H}_{k-1}}{||\hat{H}_{k-1}||^2_F + \alpha^2}
\]
2. Estimation of the point spread function

Constraints image

\[ f_k(i, j) = \begin{cases} 
  f_k(i, j) & \text{if } f_k(i, j) \in [0, 255], \\
  0 & \text{if } f_k(i, j) < 0, \\
  255 & \text{if } f_k(i, j) > 255.
\] 

The blur constraints are the nonnegativity and the normalization of the PSF

\[ \begin{cases} 
  h_k(i, j) \geq 0, \\
  \sum_{i,j} h_k(i, j) = 1.
\]
3. Kronecker product approximation

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times p$ and $s \times q$ matrices respectively. The Kronecker product of the matrices $A$ and $B$ is defined as the $(ns) \times (pq)$ matrix

$$A \otimes B = (a_{ij}B).$$

Some properties of the Kronecker product are given below.

**Lancaster & Rodman, 1995**

\[
\begin{align*}
(A \otimes B)(C \otimes D) &= (AC) \otimes (BD) \\
(A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}, \quad \text{if } A, B \text{ are nonsingular} \\
\text{vec}(AXB) &= (B^T \otimes A)\text{vec}(X) \\
(A \otimes B)^T &= A^T \otimes B^T.
\end{align*}
\]
3. Kronecker product approximation

- The PSF is usually assumed to be spatially invariant (Andrews (77), Jain (89)).
- Thus, the matrix $H$ is separable and there exist two matrices $H_1$ and $H_2$ of size $n \times n$ such that

$$H = H_2 \otimes H_1.$$ 

- The Kronecker product approximation problem (KPA)

\[ (\hat{H}_1, \hat{H}_2) = \arg \min_{H_1, H_2} \| H - H_2 \otimes H_1 \|_F. \] (5)

$\| . \|_F$ is the Frobenius norm, associated to the scalar product

$$\langle A, B \rangle_F = \text{tr}(A^T B)$$

where $\text{tr}(Z)$ denotes the trace of the square matrix $Z$ and $A$ and $B$ are two matrices in $\mathbb{R}^{n \times p}$. 
3. Kronecker product approximation


- For a given vector \( a = (a_1, \ldots, a_n)^T \in \mathbb{R}^n \),
  - The matrix \( \text{toep}(a, k) \) is an \( n \times n \) banded Toeplitz matrix whose \( k \)-th column is \( a = (a_1, \ldots, a_n)^T \).
  - The matrix \( \text{hank}(a, k) \) is an \( n \times n \) Hankel matrix with its first row and its last column defined by the vectors \( (a_{k+1}, \ldots, a_n, 0, \ldots, 0) \) and \( (0, \ldots, 0, a_1, \ldots, a_{k-1})^T \), respectively.
  - For \( a = (a_1, a_2, a_3, a_4, a_5) \), we have \( A_t = \text{toep}(a, 2) \) and \( A_h = \text{hank}(a, 3) \) with

\[
A_t = \begin{pmatrix}
    a_2 & a_1 & 0 & 0 & 0 \\
    a_3 & a_2 & a_1 & 0 & 0 \\
    a_4 & a_3 & a_2 & a_1 & 0 \\
    a_5 & a_4 & a_3 & a_2 & a_1 \\
    0 & a_5 & a_4 & a_3 & a_2
\end{pmatrix}, \quad
A_h = \begin{pmatrix}
    a_4 & a_5 & 0 & 0 & 0 \\
    a_5 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & a_1 \\
    0 & 0 & 0 & a_1 & a_2
\end{pmatrix}.
\]
3. Kronecker product approximation

- Let $P$ be an $n \times n$ matrix containing the image of the point spread function.
- Suppose the center of the PSF (location of the point source) is at $p_{ij}$.
- Let $R_n$ be the Cholesky factor of the $n \times n$ symmetric Toeplitz matrix $T_n = \text{Toeplitz}(v_n)$ with its first row $v_n = (n, 1, 0, 1, 0, \cdots)$.

**ALGORITHM: KPA**

1. Compute $R_n : T_n = R_n^T R_n$ (Cholesky decomposition)
2. Compute $P_r = R_n P R_n^T$
3. Compute the SVD: $P_r = \sum \sigma_k u_k v_k^T$
4. Construct the vectors: $\hat{a} = \sqrt{\sigma_1} R_n^{-1} v_1$ and $\hat{b} = \sqrt{\sigma_1} R_n^{-1} u_1$
5. Construct the matrices:
   - $\hat{A}_t = \text{toep}(\hat{a}, i)$, $\hat{A}_h = \text{hank}(\hat{a}, i)$,
   - $\hat{B}_t = \text{toep}(\hat{b}, j)$, $\hat{B}_h = \text{hank}(\hat{b}, j)$

It follows that $\hat{H}_1 = \hat{A}_t + \hat{A}_h$ and $\hat{H}_2 = \hat{B}_t + \hat{B}_h$ solve the following problem

$$(\hat{H}_1, \hat{H}_2) = \arg \min_{H_1, H_2} \| H - H_2 \otimes H_1 \|_F.$$
4. Tikhonov regularization

Consider the linear discrete ill-posed problem

$$\min_x \| Hx - g \|_2,$$

(6)

In order to diminish the effects of the noise in the data, we consider the Tikhonov regularization method. The method replaces the problem (6) by

$$\min_x (\| Hx - g \|_2^2 + \lambda^2 \| Lx \|_2^2),$$

(7)

where

- $L$ is a regularization operator chosen to obtain a solution with desirable properties such as small norm or good smoothness.
- Optimal value of $\lambda$: regularization parameter.
  - L-curve criterion (Hansen, SIAM 92)
  - Generalized cross-validation (GCV) method (Golub & Wahba, SIAM 77).
4. Tikhonov regularization

The problem $(13)$ is equivalent to following linear least squares problem

$$\hat{x} = \arg\min_x \left\| \begin{bmatrix} H \\ \lambda L \end{bmatrix} x - \begin{bmatrix} g \\ 0 \end{bmatrix} \right\|^2_2,$$  \hspace{1cm} (8)

The minimizer of the problem $(8)$ is computed as the solution of the following linear system

$$H_\lambda \hat{x} = H^T g,$$ \hspace{1cm} (9)

where

$$H_\lambda = H^T H + \lambda^2 L^T L.$$
4. Tikhonov regularization

We assume that $H = H_2 \otimes H_1$ and $L = L_2 \otimes L_1$ where $H_1$, $L_1$, $H_2$, $L_2$ are square matrices of dimension $n \times n$. The problem (9) can be expressed as

$$[(H_2 \otimes H_1)^T (H_2 \otimes H_1) + \lambda^2 (L_2 \otimes L_1)^T (L_2 \otimes L_1)] \hat{X} = (H_2 \otimes H_1)^T g.$$ 

then

$$(H_1^T H_1) \hat{X} (H_2^T H_2) + \lambda^2 (L_1^T L_1) \hat{X} (L_2^T L_2) = H_1^T GH_2,$$ \quad (10)

where $\hat{X}$ and $G$ are the matrices such that $\text{vec}(\hat{X}) = \hat{x}$ and $\text{vec}(G) = g$. The linear matrix equation (10) is referred to as the generalized Sylvester matrix equation and is written in the following form

$$A \hat{X} D - \lambda^2 C \hat{X} B = E,$$ \quad (11)

where

$$A = H_1^T H_1, \quad B = L_2^T L_2, \quad C = -L_1^T L_1, \quad D = H_2^T H_2, \quad E = H_1^T GH_2.$$
5. The Global-GMRES method

The method is an iterative projection method onto matrix Krylov subspaces (Bouhamidi and Jbilou, JCAM 2005, AMC 2008).

Let
\[ A_\lambda(X) = AXD - \lambda^2 CXB \]

and
\[ E = H_1^T GH_2. \]

Let \( V \) be any \( n \times n \) matrix and consider the matrix Krylov subspace associated to the pair \( (A_\lambda, V) \) and defined
\[ K_k(A_\lambda, V) = \text{span}\{ V, A_\lambda(V), \ldots, A_{\lambda}^{k-1}(V) \}. \]

We note that \( A_\lambda^i(R_0) \) is defined recursively as
\[ A_\lambda^i(R_0) = A_\lambda(A_\lambda^{i-1}(R_0)). \]

Remark that the matrix Krylov subspace \( K_k(A_\lambda, V) \) is a subspace of \( \mathbb{R}^{n \times n} \).
5. The Global-GMRES method

The modified global Arnoldi algorithm constructs an F-orthonormal basis \( V_1, V_2, \ldots, V_k \) of the matrix Krylov subspace \( \mathcal{K}_k(A, V) \), i.e.

\[
\langle V_i, V_j \rangle_F = \delta_{i,j}, \quad \text{for} \quad i, j = 1, \ldots, k,
\]

**ALGORITHM 1: The modified Global Arnoldi algorithm**

1. Set \( V_1 = V / \| V \|_F \).
2. For \( j = 1, \ldots, k \) do
   
   \( \tilde{V} = A_\lambda (V_j), \)
   
   for \( i = 1, \ldots, j \) do
   
   \[ h_{i,j} = \langle V_i, \tilde{V} \rangle_F, \]
   
   \( \tilde{V} = \tilde{V} - h_{i,j} V_i, \)
   
   endfor
   
   \[ h_{j+1,j} = \| \tilde{V} \|_F, \]
   
   \( V_{j+1} = \tilde{V} / h_{j+1,j}. \)
   
   EndFor.
5. The Global-GMRES method

The restarted Global GMRES algorithm for solving the linear matrix equation (11) is summarized as follows

<table>
<thead>
<tr>
<th>GL-GMRES algorithm for the linear matrix equation (11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Choose $X_0$ a tolerance $\varepsilon$ and set $iter = 0$. Compute: $R_0 = C - A_\lambda(X_0)$, $\beta = |R_0|_F$, and $V_1 = R_0/\beta$.</td>
</tr>
<tr>
<td>2. Construct the $F$-orthonormal basis $V_1, V_2, \ldots, V_k$ by Algorithm 1.</td>
</tr>
<tr>
<td>3. Determine $y_k$ as solution of the least square problem: $\min_{y \in \mathbb{R}^k} |R_0 e_1 - \tilde{H}_k y|_2$</td>
</tr>
<tr>
<td>Compute: $X_k = X_0 + V_k(y_k \otimes I_p)$</td>
</tr>
<tr>
<td>4. Compute the residual $R_k$ and $|R_k|_F$.</td>
</tr>
<tr>
<td>5. If $|R_k|_F &lt; \varepsilon$ Stop; else $X_0 = X_k, R_0 = R_k, \beta = |R_0|_F, V_1 = R_0/\beta, iter = iter + 1$, Goto 2.</td>
</tr>
</tbody>
</table>

$\tilde{H}_k$ denotes the $(k + 1) \times k$ upper Hessenberg matrix whose nonzero entries $h_{i,j}$ are defined by Algorithm 1.
Generalized cross-validation (GCV) method (Golub & Wahba, SIAM 77). The regularization parameter is chosen to minimize the GCV function

\[
\text{GCV}(\lambda) = \frac{\|H\hat{x}_\lambda - g\|_2^2}{\left[\text{tr}(I - HH_\lambda^{-1}H^T)\right]^2} = \frac{\|(I - HH_\lambda^{-1}H^T)g\|_2^2}{\left[\text{tr}(I - HH_\lambda^{-1}H^T)\right]^2}
\]

where \( H_\lambda = H^TH + \lambda^2L^TL \).

We have \( H = H_2 \otimes H_1 \) and \( L = L_2 \otimes L_1 \) where \( H_1, H_2L_1, L_2 \) are of size \( n \times n \).
6. The GCV method for Tikhonov regularization

Consider the Generalized Singular Value Decompositions (GSVD) (Golub & VanLoan) of the pairs \((H_1, L_1)\) and \((H_2, L_2)\). Thus, there exist orthonormal matrices \(U_1, U_2, V_1, V_2\) and invertible matrices \(X_1, X_2\) such that

\[
U_1^T H_1 X_1 = C_1 = \text{diag}(c_{1,1}, \ldots, c_{n,1}), \quad c_{i,1} \geq 0,
\]
\[
U_2^T H_2 X_2 = C_2 = \text{diag}(c_{1,2}, \ldots, c_{p,2}), \quad c_{i,2} \geq 0,
\]

\[
V_1^T L_1 X_1 = S_1 = \text{diag}(s_{1,1}, \ldots, s_{n,1}), \quad s_{i,1} \geq 0,
\]
\[
V_2^T L_2 X_2 = S_2 = \text{diag}(s_{1,2}, \ldots, s_{p,2}), \quad s_{i,2} \geq 0.
\]

and

\[
C_1^T C_1 + S_1^T S_1 = I_n, \quad C_2^T C_2 + S_2^T S_2 = I_n.
\]

Then the GSVD of the pair \((H, L)\) is given by

\[
U^T HX = C = \text{diag}(c_1, \ldots, c_{n^2}), \quad c_i \geq 0,
\]
\[
V^T LX = S = \text{diag}(s_1, \ldots, s_{n^2}), \quad s_i \geq 0,
\]

where \(U = U_2 \otimes U_1, V = V_2 \otimes V_1, C = C_2 \otimes C_1, S = S_2 \otimes S_1\) and \(X = X_2 \otimes X_1\).
Therefore, one can show that the expression of the GCV function is given by

\[
GCV(\lambda) = \frac{\sum_{i=1}^{n^2} \left( \frac{s_i^2 \tilde{g}_i}{c_i^2 + \lambda^2 s_i^2} \right)^2}{\left( \sum_{i=1}^{n^2} \frac{s_i^2}{c_i^2 + \lambda^2 s_i^2} \right)^2},
\]

where \( \tilde{g} = U^T g \).
7. Convex optimization problem

We consider the convex optimization problem

\[
\min_{x \in \tilde{\Omega}} (\|Hx - g\|_2^2 + \lambda^2 \|Lx\|_2^2),
\]

where The set \( \tilde{\Omega} \subset \mathbb{R}^{n^2} \) could be a simple convex set (e.g., a sphere or a box) or the intersection of some simple convex sets. Then

\[
\min_{x \in \Omega} (\|(H_2 \otimes H_1)x - g\|_2^2 + \lambda^2 \|(L_2 \otimes L_1)x\|_2^2),
\]

Using some Kronecker properties, the problem (14) can be reformulated as

\[
\min_{x \in \Omega} (\|H_1 X H_2^T - G\|_F^2 + \lambda^2 \|L_1 X L_2^T\|_F^2),
\]

where the set \( \Omega \) is such that

\[
x = \text{vec}(X) \in \tilde{\Omega} \subset \mathbb{R}^{n^2} \iff X \in \Omega \subset \mathbb{R}^{n \times n}.
\]
7. Convex optimization problem

\[ f_\lambda : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \]
\[ X \rightarrow \| H_1 X H_2^T - G \|_F^2 + \lambda^2 \| L_1 X L_2^T \|_F^2. \]

The convex constrained minimization problem is

\[
\text{Minimize } f_\lambda(X) \quad \text{subject to } \quad X \in \Omega. \tag{16}
\]

Specific cases that will be considered are

\[
\Omega_1 = \{ X \in \mathbb{R}^{n \times n} : L \leq X \leq U \} \tag{17}
\]

and

\[
\Omega_2 = \{ X \in \mathbb{R}^{n \times n} : \| X \|_F \leq \delta \}. \tag{18}
\]

Here, \( Y \leq Z \) means \( Y_{ij} \leq Z_{ij} \) for all possible entries \( ij \), \( L \) and \( U \) are given matrices and \( \delta > 0 \) is a given scalar. Another option to be considered is \( \Omega = \Omega_1 \cap \Omega_2 \).

The function \( f_\lambda : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \) is differentiable and its gradient is given by

\[
\nabla f_\lambda(X) = 2 \left( H_1^T (A(X) - G) H_2 + \lambda^2 L_1^T L(X) L_2 \right),
\]
\[
= 2 \left( H_1^T (H_1 X H_2^T - G) H_2 + \lambda^2 L_1^T L_1 X L_2^T L_2 \right). 
\]
7. Convex optimization problem

Let

\[
[X_k]_{ij} = \begin{cases} 
L_{ij} & \text{if } [\nabla f_\lambda(X_k)]_{ij} \geq 0, \\
U_{ij} & \text{if } [\nabla f_\lambda(X_k)]_{ij} < 0,
\end{cases}
\]  

and

\[
\alpha_k = -\frac{\langle \mathcal{A}(X_k) - G| \mathcal{A}(H_k) \rangle_F + \lambda^2 \langle \mathcal{L}(X_k)| \mathcal{L}(H_k) \rangle_F}{\| \mathcal{A}(H_k) \|^2_F + \lambda^2 \| \mathcal{L}(H_k) \|^2_F}.
\]
## 7. Convex optimization problem

### Algorithm 2: The Conditional Gradient-Tikhonov Algorithm

1. Choose a tolerance $tol$, an initial guess $X_0 \in \Omega$, an integer $k_{\text{max}}$ and set $k = 0$.
2. Determine $\lambda$ as the parameter minimizing the GCV function $G(\lambda)$ given by equation (12).
3. While $k < k_{\text{max}}$
   
   3.1- Compute the matrix $\overline{X}_k$ by using (19),
   
   3.2- Compute the value: $\eta_k = \langle \nabla f_\lambda(X_k) | \overline{X}_k - X_k \rangle_F$
   
   3.3- If $|\eta_k| < tol$ Stop else continue,
   
   3.4- Compute $\alpha_k$ by using (20),
   
   3.5- If $\alpha_k > 1$ then $\alpha^*_k = 1$,
   
   ElseIf $\alpha_k < 0$ then $\alpha^*_k = 0$,
   
   Else $\alpha^*_k = \alpha_k$,
   
   EndIF.
   
   3.6- Update $X_{k+1} = X_k + \alpha^*_k(\overline{X}_k - X_k)$,
   
   3.7- Set $k = k + 1$,

EndWhile.
7. Convex optimization problem

Theorem

The sequence \( \{X_k\} \) generated by the Algorithm 2 is a minimizing sequence, i.e.,

\[
\lim_{k \to \infty} f_\lambda(X_k) = \min_{X \in \Omega} f_\lambda(X).
\]
8. Numerical examples

The PSNR is the Peak Signal-to-Noise Ratio (PSNR) and it measures the distortion between the original image $I_0$ and another image $I_r$ (restored image) and is defined by

$$PSNR(I_0, I_r) = 10 \log_{10}\left(\frac{n^2d^2}{\|I_0 - I_r\|^2_F}\right),$$

where $d = 255$ in the case of gray images and $n \times n$ is the size of the images.
8. Numerical examples: Tikhonov-Sylvester method

The original image is the cameraman image from Matlab

The blurring matrix $H$ is given by $H = H_2 \otimes H_1 \in \mathbb{R}^{256^2 \times 256^2}$, where $H_1 = H_2 = [h_{ij}]$ and $[h_{ij}]$ is the Toeplitz matrix of dimension $256 \times 256$ given by

$$h_{ij} = \begin{cases} \frac{1}{2r-1}, & |i - j| \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

**Figure:** Original image (left), degraded image (right).
8. Numerical examples: Tikhonov-Sylvester method

Figure: Degraded image (left) and restored image (right).
8. Numerical examples: Conditional gradient Tikhonov method

Figure: Original image

The original 500 × 500 "cat" image was degraded by a 'speckle' multiplicative noise with different values of the variance $\sigma_m$ plus an additive white gaussian noise with zero mean and different values of the variance $\sigma_a$. 
8. Numerical examples: Conditional gradient Tikhonov method

Figure: Left: degraded image with $PSNR = 16.02\, dB$, $\sigma_m = 0.1$ and $\sigma_a = 0.001$, right: degraded image with $PSNR = 13.88\, dB$, $\sigma_m = 0.25$ and $\sigma_a = 0.001$
8. Numerical examples: Conditional gradient Tikhonov method

**Figure:** Left: Restored image by `deconvlucy` of Matlab with $PSNR = 14.27\, dB$, right: Restored image by our method $PSNR = 23.53\, dB$ with $\sigma_m = 0.1$ and $\sigma_a = 0.001$
8. Numerical examples: Conditional gradient Tikhonov method

Figure: Left: Restored image by deconvlucy of Matlab with $PSNR = 12.65\, dB$, right: Restored image by our method $PSNR = 20.69\, dB$ with $\sigma_m = 0.25$ and $\sigma_a = 0.001$
8. Numerical examples: Conditional gradient Tikhonov method

<table>
<thead>
<tr>
<th>$\sigma_m$</th>
<th>$\sigma_a$</th>
<th>Degraded image</th>
<th>deconvlucy</th>
<th>CGT method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.001</td>
<td>16.02</td>
<td>14.27</td>
<td>23.53</td>
</tr>
<tr>
<td>0.25</td>
<td>0.001</td>
<td>13.88</td>
<td>12.65</td>
<td>20.69</td>
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<tr>
<td>0.50</td>
<td>0.001</td>
<td>11.83</td>
<td>10.13</td>
<td>18.99</td>
</tr>
<tr>
<td>0.001</td>
<td>0.1</td>
<td>14.54</td>
<td>13.22</td>
<td>18.36</td>
</tr>
<tr>
<td>0.001</td>
<td>0.25</td>
<td>12.00</td>
<td>09.66</td>
<td>15.03</td>
</tr>
</tbody>
</table>

**Table:** PSNR for different values of the variance of the multiplicative and the additive noises
9. References


